The Gamma function.

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Weierstrass Δ function.

$$\Delta(z) := e^{\gamma z} G(z) = \lim_{n\to\infty} \frac{z(z+1)\dots(z+n)}{n!\, n^z} = e^{\gamma z} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}.$$

Properties. 1) $\overline{\Delta(z)} = \Delta(\bar z)$

2) $\forall x>0\ (x\in\mathbb{R})$: $\Delta(x)>0$

3) $\Delta(1)=1$.

4) $\Delta(z) = z\Delta(z+1)$

Proof $\Delta(z+1) = \lim_{n\to\infty} \frac{(z+1)\dots(z+n+1)}{n!\, n^{z+1}} = \Big(\underbrace{\lim_{n\to\infty}\frac{z(z+1)\dots(z+n)}{n!\, n^z}}_{\Delta(z)}\Big)\Big(\underbrace{\lim_{n\to\infty}\frac{(z+n+1)}{z\cdot n}}_{\frac{1}{z}}\Big) = \frac{\Delta(z)}{z}.$

5) $\Delta(z)\Delta(1-z) = \frac{\sin\pi z}{z}$

Proof. $\Delta(z)\Delta(1-z) \underset{\text{by } 4)}{=} \Delta(z)\cdot\left(-\frac{1}{z}\Delta(-z)\right) = -\frac{1}{z}\Delta(z)\Delta(-z) = -\frac{1}{z}G(z)G(-z) = \frac{\sin\pi z}{\pi}$ ■

Gamma Function

Def. $\Gamma(z) := \frac{1}{\Delta(z)} = e^{-\gamma z}\frac{1}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n} = \lim_{n\to\infty}\frac{n!\, n^z}{z(z+1)\dots(z+n)}$

Properties. 1) $\Gamma\in\mathcal{A}(\mathbb{C}\setminus\{0,-1,-2,\dots\})$, has simple poles at $\{0,-1,\dots,-n,\}$.

2) $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(1)=1$

$\left(\frac{1}{\Delta(z+1)} = \frac{z}{\Delta(z)}\right)$

By induction, $\Gamma(z+n) = z(z+1)\dots(z+n-1)\Gamma(z)$

In particular, $\underline{\Gamma(n) = (n-1)!}$, $n\in\mathbb{N}$.

3) $\operatorname{res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$

Proof Simple pole, so

$\operatorname{res}_{z=-n}\Gamma(z) = \lim_{z\to -n}(z+n)\Gamma(z) = \lim_{z\to -n}\frac{\Gamma(z+n+1)}{z(z+1)\dots(z+n-1)} =$

$\frac{\Gamma(1)}{(-n)(-n+1)\dots(-1)} = \frac{(-1)^n}{n!}$ ■

4) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$ (Euler supplement)

$\Gamma(\bar z) = \overline{\Gamma(z)}$, $x>0 \Rightarrow \Gamma(x)>0$.

In particular, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{\sin\frac{\pi}{2}}} = \sqrt{\pi}$, $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$ (induction)

5) $|\Gamma(x+iy)| \le \Gamma(x)$.

Proof. $|n^z| = n^x$, $|z+k| \ge (x+k)$, use the limit formula ■

6) $|\Gamma(iy)|^2 = \Gamma(iy)\Gamma(-iy) = \frac{\Gamma(iy)\Gamma(1-iy)}{-iy} = \frac{\pi}{iy\sin(\pi iy)} =$

$\boxed{\frac{\pi}{y\sinh\pi y}}$

$|\Gamma(\frac{1}{2}+iy)|^2 = \Gamma(\frac{1}{2}+iy)\Gamma(\frac{1}{2}-iy) = \frac{\pi}{\sin(\frac{\pi}{2}-iy)} = \frac{\pi}{\cos(iy)} = \frac{\pi}{\cosh\pi y}$..

Logarithmic derivative of Γ:

$\psi := \frac{\Gamma'}{\Gamma} \in \mathcal{A}(\mathbb{C} \setminus \{0,-1,\dots,-n,\}) \cap \mathcal{M}(\mathbb{C})$

$\psi = -\frac{\Delta'}{\Delta}$

Claim $\psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$ - uniformly on compacts.

Proof. $\Delta = e^{\gamma z} z \prod \left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$ uniformly on compacts. Can take log and use Weierstrass to differentiate ∎

Corollary. 1) $\Gamma'(1) = \psi(1) = -\gamma$, $\psi(k) = 1 + \frac{1}{2} + \dots + \frac{1}{k-1} - \gamma$, $k \in \mathbb{N}, k \geq 2$.

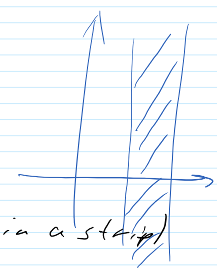
Proof. $\Gamma'(1) = -\gamma - 1 - \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n}\right) = -\gamma - 1 + 1 = -\gamma$

$$\psi(k) = -\gamma - 1 - \sum_{n=1}^{\infty}\left(\frac{1}{n+k} - \frac{1}{n}\right) = 1 + \frac{1}{2} + \dots + \frac{1}{k-1} - \gamma \ ∎$$

2) $\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$ - locally uniformly in $\mathbb{C}$.

Proof. Differentiate the series for ψ termwise ∎

Helmut Wielandt



Uniqueness Theorem (Wielandt)

Let $F \in \mathcal{A}(\{\operatorname{Re} z > 0\})$, $F(z+1) = z F(z)$.

Assume: $\exists C : \ \forall z \in \{1 \leq \operatorname{Re} z \leq 2\}$, $|F(z)| \leq C$ (bounded in a strip)

Then $F(z) = a\Gamma(z) \quad \forall z \in \{\operatorname{Re} z > 0\}$

where $a = F(1)$.

Proof. Let $v := F - a\Gamma$. $v(z+1) = z v(z)$, $v(z+n) = z(z+1)\dots(z+n-1)v(z)$

Extend v:

Let for $\operatorname{Re} z > 0$:

$v_1(z) := \frac{v(z+1)}{z}$.

Then $v_1(z) = v(z)$ for $\operatorname{Re} z > 0$.

For $\operatorname{Re} z > -n$, define $v_n(z) = \frac{v(z+n)}{z(z+1)\dots(z+n-1)}$.

Again, for $\operatorname{Re} z > 0$, $v_n(z) = v(z)$.

Define $v(z) = v_n(z)$ if $\operatorname{Re} z > -n$. Well-define, since for $\operatorname{Re} z > 0$ $v_n(z) = v_k(z)$, so, by uniqueness, $v_n(z) = v_k(z)$ if both defined.

$v(z) \in \mathcal{M}(\mathbb{C}) \cap \mathcal{A}(\mathbb{C} \setminus \{0,-1,\dots,-n,\dots\})$, v can have poles only at $\{0,-1,\dots,-n\dots\}$

$v(z+n+1)$ $v(1)$

$v(z) \in \mathcal{M}(\mathbb{C}) \cap \mathcal{A}(\mathbb{C} \setminus \{0,-1,\dots,-n,\dots\})$, v can have poles only at $\{0,-1,\dots,-n\dots\}$

But $\lim_{z\to -n} (z+n)\, v(z) = \lim_{z\to -n} \frac{v(z+n+1)}{z(z+1)\dots(z+n-1)} = \frac{v(1)}{(-n)\dots(-1)} = 0$ (since $v(1)=0$).

So v has removable singularity at $-n$, $v \in \mathcal{A}(\mathbb{C})$.

Note that Γ is bounded on $\{1 \le \operatorname{Re} z < 2\}$, since

$|\Gamma(x+iy)| \le \Gamma(x) \le \max_{x\in[1,2]} \Gamma(x)$.

So v is bounded in the strip: for some M $|v(z)| \le M$

Let us now consider $v(z)$ for $0 \le \operatorname{Re} z \le 1$.

In the square $\{0 \le \operatorname{Re} z \le 1,\ 0 \le \operatorname{Im} z \le 1\}$ v is bounded (by compactness).

For $|\operatorname{Im} z| > 1$, $|v(z)| = \frac{|v(z+1)|}{|z+1|} \le M$.

$1 \le \operatorname{Re}(z+1) \le 2$

$|z+1| \ge |\operatorname{Im} z| \ge 1$

So v is bounded in $\{0 \le \operatorname{Re} z \le 1\}$.

Finally, let $q(z) := v(z)\, v(1-z)$

Then $q(z+1) = v(z+1)\, v(-z) = z\, v(z) \cdot \frac{v(1-z)}{-z} = -q(z)$ – periodic $(q(z+2) = q(z))$

In $0 \le \operatorname{Re} z \le 1$, q is bounded $\left(\begin{array}{l} 0 \le \operatorname{Re} z \le 1 \\ 0 \le \operatorname{Re}(1-z) \le 1 \end{array}\right)$

So, by periodicity, $q \in \mathcal{A}(\mathbb{C})$

q - bounded.

So, by Liouville $v(z)\, v(1-z) = q(z) = q(1) = 0$.

So $v \equiv 0$ (if not, all zeroes are isolated, so cannot $v(z)v(1-z) \equiv 0$). $\blacksquare$

Duplication formula

$$\sqrt{\pi}\, \Gamma(2z) = 2^{2z-1}\, \Gamma(z)\, \Gamma(z+\tfrac{1}{2})$$

Proof. Let $F(z) := \dfrac{\Gamma(\frac{z}{2})\, \Gamma(\frac{z}{2}+\frac{1}{2})}{2^{\frac{1}{2}-z}\sqrt{2\pi}}$. Then

$$F(z+1) = \frac{\Gamma(\frac{z+1}{2})\, \Gamma(\frac{z}{2}+1)}{2^{-z-\frac{1}{2}}\sqrt{2\pi}} = \frac{\Gamma(\frac{z+1}{2})\, \Gamma(\frac{z}{2}) \cdot \frac{z}{2}}{2^{-z-\frac{1}{2}}\sqrt{2\pi}} = z\, F(z).$$

$$F(1) = \frac{\Gamma(\frac{1}{2})\,\Gamma(1)}{\sqrt{\pi}} = 1.$$

And if $1 < \operatorname{Re} z \le 2$, $\left|2^{z-\frac{1}{2}}\right| \le 2^{2-\frac{1}{2}}$, and both $\Gamma(\frac{z}{2})$, $\Gamma(\frac{z}{2}+\frac{1}{2})$ are bounded.

So by Wielandt's Theorem, $\Gamma(z) = F(z) = \dfrac{\Gamma(\frac{z}{2})\, \Gamma(\frac{z}{2}+\frac{1}{2})}{2^{\frac{1}{2}-z}\sqrt{2\pi}}$

Euler integral formula for Γ-function

Theorem. If $\operatorname{Re} z > 0$ then

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

Proof. Let $\varphi_n(z) := \int_{1/n}^{n} t^{z-1} e^{-t}\, dt$.

Then, since $\dfrac{t^{z+h-1} - t^{z-1}}{h} \xrightarrow[h\to 0]{} \log t\; t^{z-1}$ uniformly on $[\frac{1}{n}, n]$, for any z.

$$\frac{\varphi_n(z+h) - \varphi_n(z)}{h} = \int_{1/n}^{n} \frac{t^{z+h-1} - t^{z-1}}{h} e^{-t}\, dt \xrightarrow[h\to 0]{} \int_{1/n}^{n} \log t\; t^{z-1} e^{-t}\, dt,$$

$\xrightarrow[h\to 0]{}$ uniformly on $[\frac{1}{n}, n]$, for any t.

$$\frac{\varphi_n(z+h)-\varphi_n(z)}{h} = \int_{1/n}^{n} \frac{t^{z+h-1}-t^{z-1}}{h} e^{-t}\,dt \xrightarrow[h\to 0]{} \int_{1/n}^{n} \log t\, t^{z-1} e^{-t}\,dt,$$

So $\varphi_n(z) \in \mathcal{A}(\mathbb{C})$.

Let $\varphi(z) := \int_0^\infty t^{z-1} e^{-t}\,dt$.

Then $$|\varphi_n(z)-\varphi(z)| = \left|\int_0^{1/n} + \int_n^\infty t^{z-1}e^{-t}\,dt\right| \le \underbrace{\int_0^{1/n} t^{\operatorname{Re} z-1}\,dt}_{\mathrm{I}} + \underbrace{\int_n^\infty t^{\operatorname{Re} z-1}e^{-t}\,dt}_{\mathrm{II}}.$$

$\mathrm{I} \xrightarrow[n\to\infty]{} 0$ uniformly in $\operatorname{Re} z > \delta$

$\mathrm{II} \xrightarrow[n\to\infty]{} 0$ uniformly in $\operatorname{Re} z < r$

So $\varphi_n \to \varphi$ locally uniformly in $\{\operatorname{Re} z > 0\}$, $\varphi \in \mathcal{A}(\{\operatorname{Re} z > 0\})$.

For $x > 0$, $$\varphi(x+1) = \int_0^\infty t^x e^{-t}\,dt = -\int_0^\infty t^x\, de^{-t} = t^x e^{-t}\Big|_0^\infty + \int_0^\infty x t^{x-1} e^{-t}\,dt \;=\; x\int_0^\infty t^{x-1}e^{-t}\,dt = x\,\varphi(x).$$

So by uniqueness for analytic functions $\varphi(z+1) = z\,\varphi(z)$.

Also, when $1 \le \operatorname{Re} z \le 2$, $|\varphi(z)| \le \int_0^1 e^{-t}\,dt + \int_1^\infty t e^{-t}\,dt$ – bounded.

$|t^{z-1}| \le 1$ when $\operatorname{Re} z \ge 1$, $t \le 1$ $\qquad \varphi(1) = \int_0^\infty e^{-t}\,dt = 1$.

$|t^{z-1}| \le t$ when $\operatorname{Re} z \le 2$, $t \ge 1$.

By Wielandt's uniqueness theorem,

$$\varphi(z) = \Gamma(z). \blacksquare$$

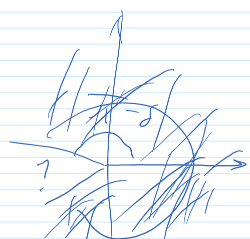
Stirling formula

For $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} e^{\mu(z)}$$

where $|\mu(z)| \le \frac{1}{8} \frac{1}{\cos^2(\varphi/2)} \frac{1}{|z|}$, $\operatorname{Arg} z = \varphi$

$|\mu(z)| \le \frac{1}{8}\frac{1}{\sin^2(\delta/2)}\frac{1}{|z|}$, $|\operatorname{Arg} z| \le \pi - \delta$.



Corollary (Classical Stirling Formula).

$$n! = n\Gamma(n) = \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\mu(n)}, \text{ where } |\mu(n)| \le \frac{1}{8n}.$$

Claim Let $\operatorname{Arg} z = \varphi$, $t > 0$. Then

$$|z+t| \ge (|z|+t)\cos\frac{\varphi}{2}.$$

Proof. $$|z+t|^2 = (z+t)(\bar z + t) = |z|^2 + t^2 + 2\underbrace{\operatorname{Re} zt}_{t|z|\cos\varphi} = (|z|+t)^2 - 2|z|t(1-\cos\varphi)$$

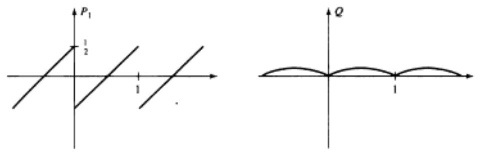
$$(|z|+t)^2 - 4|z|t\sin^2\frac{\varphi}{2} \ge (|z|+t)^2 - (|z|+t)^2\sin^2\frac{\varphi}{2} = (|z|+t)^2\cos^2\frac{\varphi}{2} \blacksquare$$

In particular, when $|\operatorname{Arg} z| \le \pi - \delta$, $\cos^2\frac{\varphi}{2} \le \cos^2\frac{\pi-\delta}{2} = \sin^2\frac{\delta}{2}$.

Define now $P_1(t) := t - \lfloor t \rfloor - \frac{1}{2}$

$Q(t) := \frac{1}{2}\left(t - \lfloor t\rfloor - (t - \lfloor t \rfloor)^2\right)$

Here $\lfloor t \rfloor = \max\{n \in \mathbb{Z} : n \le t\}$.



Observe: $P_1(t+1) = P_1(t)$

$Q(t+1) = Q(t)$

$Q'(t) = -P_1(t)$, $t \notin \mathbb{Z}$, $Q(t) \geq 0$

$$\max_{t \in \mathbb{R}} Q(t) = \max_{0 \leq t < 1} Q(t) = \max_{0 \leq t < 1} \frac{t - t^2}{2} = \frac{1}{8}.$$

Define: $\mu(z) := -\int_0^\infty \frac{P_1(t)\,dt}{z+t} = \int_0^\infty \frac{Q(t)}{(z+t)^2}\,dt \in A(\mathbb{C} \setminus (-\infty, 0])$.

Proof (convergence and analyticity).

Let $\pi \geq \delta > 0$. Then if $|\operatorname{Arg} z| \leq \pi - \delta$, and $|z| > \varepsilon$; $|z+t|^2 \geq \sin^2 \frac{1}{2}\delta$.

So $\left|\frac{Q(t)}{(z+t)^2}\right| \leq \frac{1}{8} \frac{1}{\sin^2 \frac{1}{8}} \frac{1}{(\varepsilon + t)^2}$

So $\lim_{n \to \infty} \int_0^n \frac{Q(t)}{(z+t)^2}\,dt$ converges locally uniformly in $\mathbb{C} \setminus (-\infty, 0]$.

So $\int_0^\infty \frac{Q(t)}{(z+t)^2}\,dt \in A(\mathbb{C} \setminus (-\infty, 0])$.

But $\int_0^\infty \frac{Q(t)}{(z+t)^2}\,dt = \sum_{n=0}^\infty \int_n^{n+1} \frac{Q(t)}{(z+t)^2}\,dt \overset{\text{by Parts}}{=} \sum_{n=0}^\infty \left(-\frac{Q(t)}{(z+t)}\Big|_n^{n+1} + \int_n^{n+1} \frac{Q'(t)\,dt}{(z+t)} \right) =$

$$= \sum_{n=0}^\infty -\int_n^{n+1} \frac{P_1(t)\,dt}{z+t} = -\int_0^\infty \frac{P_1(t)\,dt}{z+t} \quad \blacksquare$$

Claim $\mu(z) - \mu(z+1) = \int_0^1 \frac{\frac{1}{2} - t}{z+t}\,dt = (z + \frac{1}{2}) \operatorname{Log}(1 + \frac{1}{z}) - 1$, $z \notin (-\infty, 0]$.

Proof. $\mu(z+1) = -\int_0^\infty \frac{P_1(t+1)}{z+t+1}\,dt \underset{t := t+1}{=} -\int_1^\infty \frac{P_1(t)}{z+t}\,dt = \mu(z) - \int_0^1 \frac{\frac{1}{2} - t}{z+t}\,dt =$

$P_1(t+1) = P_1(t)$

$\mu(z) - \left((z + \frac{1}{2}) \operatorname{Log}(z+t) - t\right)\Big|_{t=0}^{t=1} = \mu(z) - (z + \frac{1}{2})(\operatorname{Log}(z+1) - \operatorname{Log} z) + 1 =$

$\mu(z) - (z + \frac{1}{2}) \operatorname{Log}(1 + \frac{1}{z}) + 1 \quad \blacksquare$

Claim. (1) $|\mu(z)| \leq \frac{1}{8} \frac{1}{\cos^2 \frac{\varphi}{2}} \frac{1}{|z|}$ $\operatorname{Arg} z = \varphi$

(2) $|\mu(z)| \leq \frac{1}{8} \frac{1}{\sin^2 \frac{\delta}{2}} \frac{1}{|z|}$ $|\operatorname{Arg} z| \leq \pi - \delta$.

Proof. (2) $\Leftarrow$ (1) $\left(\sin^2\left(\frac{\delta}{2}\right) = \cos^2\left(\frac{\pi - \delta}{2}\right) \geq \cos^2 \frac{\varphi}{2} \right.$ if $\left. |\varphi| \leq \pi - \delta \right)$.

$Q(t) \leq \frac{1}{8}$, so, since $|z+t|^2 \geq (|z| + t)\cos^2 \frac{\varphi}{2}$:

$$|\mu(z)| = \left|\int_0^\infty \frac{Q(t)}{(z+t)^2}\,dt\right| \leq \frac{1}{8} \int_0^\infty \frac{1}{(|z| + t)^2 \cos^2 \frac{\varphi}{2}}\,dt = \frac{1}{8|z| \cos^2 \frac{\varphi}{2}} \underbrace{\int_0^\infty \frac{ds}{(1+s)^2}}_{=1} \quad \blacksquare$$

$s = t/|z|$

Proof of Stirling

Let $\Phi(z) = \sqrt{2\pi}\, z^{z - \frac{1}{2}} e^{-z} e^{\mu(z)} \in A(\mathbb{C} \setminus (-\infty, 0])$.

$$\Phi(z+1) = \sqrt{2\pi}\ (z+1)^{z+\frac{1}{2}}\ e^{-z-1}\ e^{\mu(z+1)} = \sqrt{2\pi}\ (z+1)^{z+\frac{1}{2}} e^{-z-1}\ e^{\mu(z)} \cdot e \cdot e^{(z+\frac{1}{2})\operatorname{Log}(1+\frac{1}{z})} = \boxed{z\,\Phi(z)}\ .$$

Let us now prove that Φ is bounded in the strip $\{1 \le \operatorname{Re} z \le 2\}$.

$e^{\mu(z)}$ is bounded (since $\mu(z)$ is bounded, by $\frac{1}{8\sin^2\frac{\pi}{4}}$, since $|z| \ge 1$, $\operatorname{Arg} z \le \frac{\pi}{2}$)

If $z = x+iy = |z|e^{i\varphi}$, we have

$$|z^{z-\frac{1}{2}} e^{-z}| = |e^{(\log|z| + i\varphi)(x-\frac{1}{2}+iy)} \cdot e^{-x-iy}| = |z|^{x-\frac{1}{2}} e^{-x-\varphi y}$$

If $1 < x \le 2$, $|y| \ge 2 \Rightarrow x - \frac{1}{2} < 2$, $|z| = \sqrt{x^2+y^2} \le 2y$

$-y\varphi \le -\frac{\pi}{2}|y|$

so $|z^{z-\frac{1}{2}} e^{-z}| \le 4y^2 e^{-\frac{1}{2}\pi|y|} \le C.$

If $1 < x \le 2$, $|y| \le 2$ – Φ is bounded by compactness.

So, by Wielandt's Theorem,

$\Gamma(z) = a\,\Phi(z)$ for some a.

Now use duplication formula:

$$\Gamma(2z) = a\,\Phi(2z) = \frac{2^{2z-1}\Gamma(z)\ \Gamma(z+\frac{1}{2})}{\sqrt{\pi}} = \frac{a^2\ \Phi(z)\Phi(z+\frac{1}{2})\ 2^{2z-1}}{\sqrt{\pi}}$$

$$\sqrt{e}\, e^{\mu(2z) - \mu(z) - \mu(z+\frac{1}{2})} = a\left(1+\tfrac{1}{2z}\right)^z$$

Take $z = x > 0$ and let $x \to \infty$. $\mu(2x) \to 0$, $\mu(x) \to 0$, $\mu(x+\frac{1}{2}) \to 0$, $(1+\frac{1}{2x})^x \to \sqrt{e}$.

So $a = 1$ ∎